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Quasi-plane waves for a particle with spin $1/2$ on the background of Lobachevsky geometry: simulating of a special medium

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Abstract

In the paper complete systems of exact solutions for Dirac and Weyl equations in the Lobachevsky space H_3 are constructed on the base of the method of separation of the variables in quasi-cartesian coordinates. An extended helicity operator is introduced. It is shown that solution constructed when translating to the limit of vanishing curvature coincide with common plane wave solutions on Minkowski space going in opposite z -directions. It is shown the problem posed in Lobachevsky space simulates a situation in the flat space for a quantum-mechanical particle of spin $1/2$ in a 2-dimensional potential barrier smoothly rising to infinity on the right.

It is known that in the field theory of elementary particles, the basis of plane wave is of the most use. However, in presence of a curvature, any common plane wave solutions do not exist. Therefore, of a special interest are examples non-Euclidean spaces in which some analogues of such solutions can be constructed. In the paper [1], it was shown that in the Lobachevsky space there are such solutions for particles with spin 0; also see the books by Gelfand–Graev–Vilenkin [4], [5]. An analog of plane waves in a space of constant positive curvature was studied by Volobuev [2]. The later treatment of this problem was given in [3]. Solutions of the plane wave type for Maxwell’s equations have been considered in [6]–[9]. In [10], the

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problem of constructing solutions of the Dirac equation in the Lobachevsky space was studied on the base of the method of squaring; in particular, it was pointed out the possibility of constructing solutions of the Dirac plane wave studied starting with Shapiro's scalar waves . In this paper we will construct a complete basis of solutions of the plane wave type for Dirac and Weyl particles in the Lobachevsky space, applying the method of separation of the variables in a special system of quasi-cartesian coordinates closely related to horospherical coordinates.

To understand the physical meaning of the system under consideration, it should be mentioned that Lobachevsky geometry simulates a medium with special constitutive relations. The situation is specified in quasi-cartesian coordinates (x, y, z) was treated in [9]. Exact solutions of the Maxwell equations in complex 3-vector form, extended to curved space models within the tetrad formalism, have been found in Lobachevsky space. The problem reduces to a second order differential equation which can be associated with an 1-dimensional Schrödinger problem for a particle in external potential field $U(z) = U_0 e^{2z}$. In quantum mechanics, curved geometry acts as an effective potential barrier with reflection coefficient $R = 1$; in electrodynamic context results similar to quantum-mechanical ones arise: the Lobachevsky geometry simulates a medium that effectively acts as an ideal mirror. Penetration of the electromagnetic field into the effective medium, depends on the parameters of an electromagnetic wave, frequency ω , $k_1^2 + k_2^2$, and the curvature radius ρ .

In the present paper, that analysis will be extended to the case of particles with spin 1/2, described by equations of Dirac and Weyl. The generalized spinor plane waves can find application in the analysis of the behavior of fermions particles on cosmological scales, or in simulating special media affecting the spinor particles.

1 On the solutions of the Schrödinger equation

In the Lobachevsky space-time parameterized by quasi-cartesian coordinates

$$dS^2 = dt^2 - e^{-2z}(dx^2 + dy^2) - dz^2 ;$$

the element of volume is given by

$$dV = \sqrt{-g} dx dy dz = e^{-2z} dx dy dz , \quad x, y, z \in (-\infty, +\infty) .$$

The magnitude and sign of the z are substantial, in particular when referring to the probabilistic interpretation of the wave functions

$$dW = |\Psi|^2 dV = |\Psi|^2 e^{-2z} dx dy dz .$$

Let us describe some details of the parametrization of the space by coordinates (x, y, z) . It is known that this model can be identified with a branch of hyperboloid in 4-dimension flat space

$$u_0^2 - u_1^2 - u_2^2 - u_3^2 = \rho^2, \quad u_0 = +\sqrt{\rho^2 + \mathbf{u}^2}.$$

Coordinates x, y, z are referred to u_a by relations

$$\begin{aligned} u_1 &= xe^{-z}, \quad u_2 = ye^{-z}, \\ u_3 &= \frac{1}{2}[(e^z - e^{-z}) + (x^2 + y^2)e^{-z}], \\ u_0 &= \frac{1}{2}[(e^z + e^{-z}) + (x^2 + y^2)e^{-z}]. \end{aligned} \quad (1.1a)$$

It is convenient to employ 3-dimensional Poincaré realization for Lobachevsky space as an inside part of 3-sphere:

$$q_i = \frac{u_i}{u_0} = \frac{u_i}{\sqrt{\rho^2 + u_1^2 + u_2^2 + u_3^2}}, \quad q_i q_i < +1. \quad (1.1b)$$

Quasi-cartesian coordinates (x, y, z) are referred to q_i as follows

$$\begin{aligned} q_1 &= \frac{2x}{x^2 + y^2 + e^{2z} + 1}, \quad q_2 = \frac{2y}{x^2 + y^2 + e^{2z} + 1}, \\ q_3 &= \frac{x^2 + y^2 + e^{2z} - 1}{z^2 + y^2 + e^{2z} + 1}. \end{aligned} \quad (1.1c)$$

Inverses to (1.1c) relations are

$$x = \frac{q_1}{1 - q_3}, \quad y = \frac{q_2}{1 - q_3}, \quad e^z = \frac{\sqrt{1 - q_3^2}}{1 - q_3}. \quad (1.1d)$$

In particular, note that on the axis $q_1 = 0, q_2 = 0, q \in (-1, +1)$ relations (1.1d) assume the form

$$x = 0, \quad y = 0, \quad e^z = \sqrt{\frac{1 + q_3}{1 - q_3}},$$

that is

$$\begin{aligned} q_3 &\longrightarrow +1, & e^z &\longrightarrow +\infty, & z &\longrightarrow +\infty; \\ q_3 &\longrightarrow -1, & e^z &\longrightarrow +0, & z &\longrightarrow -\infty. \end{aligned} \quad (1.2)$$

Schrödinger equation in Riemannian space [11]

$$i \hbar \partial_t \Psi = \frac{1}{2M} \left[\left(\frac{i \hbar}{\sqrt{-g}} \partial_k \sqrt{-g} + e A_k \right) (-g^{kl}) (i \hbar \partial_l + e A_l) \right] \Psi$$

in quasi-cartesian coordinates (1.1a) takes the form

$$i \hbar \frac{\partial}{\partial t} \Psi = -\frac{\hbar^2}{2M} \left[e^{2z} \frac{\partial^2}{\partial x^2} + e^{2z} \frac{\partial^2}{\partial y^2} + e^{2z} \frac{\partial}{\partial z} e^{-2z} \frac{\partial}{\partial z} \right] \Psi.$$

The variables are separated by the substitution $\Psi = e^{-iEt/\hbar} e^{ik_1 x} e^{ik_2 y} f(z)$:

$$\left[\frac{d^2}{dz^2} - 2 \frac{d}{dz} + \epsilon - e^{2z} (k_1^2 + k_2^2) \right] f(z) = 0, \quad (1.3a)$$

where a dimensionless quantity used $\epsilon = 2ME\rho^2/\hbar^2$, ρ – curvature radius of the space. Elementary substitution $f = e^z \varphi(z)$ in equation (1.3a) gives a Schrödinger-like equation

$$\left(\frac{d^2}{dz^2} + \epsilon - 1 - (k_1^2 + k_2^2) e^{2z} \right) \varphi(z) = 0 \quad (1.3b)$$

with potential function

$$U(z) = 1 + (k_1^2 + k_2^2) e^{2z}. \quad (1.3c)$$

Note that the probabilistic interpretation of the wave function after the transformation to φ reads

$$dW = |\Psi|^2 dV = |\varphi|^2 dx dy dz. \quad (1.4)$$

An easily interpretable physical solution for $\epsilon > 1$ is the following: on the left we have the superposition of two waves, falling from the left and reflected. On the right behind the barrier, the wave function must sharply decrease to zero.

It should be noted that the case $k_1 = 0$, $k_2 = 0$ is special: the equation (1.3a) is very much changed because the potential function disappears

$$\left(\frac{d^2}{dz^2} - 2 \frac{d}{dz} + \epsilon \right) f(z) = 0, \quad f = e^{(1 \pm i\sqrt{\epsilon-1})z}, \quad \varphi = e^{(\pm i\sqrt{\epsilon-1})z}, \quad (1.5)$$

and the function φ is a solutions of the type of ordinary plane wave.

Let us turn to the general case and in eq. (1.3a) introduce the variable

$$\sqrt{k_1^2 + k_2^2} e^z = Z, \quad Z \in (0, +\infty);$$

the equation takes the form

$$\left(\frac{d^2}{dZ^2} - \frac{1}{Z} \frac{d}{dZ} + \frac{\epsilon}{Z^2} - 1 \right) f(Z) = 0; \quad (1.6)$$

with the help of a substitution $f = \sqrt{Z} F$, one can remove the term with the first derivative

$$\left(\frac{d^2}{dZ^2} + \frac{\epsilon - 3/4}{Z^2} - 1 \right) F(Z) = 0.$$

This form makes it easy to find the asymptotical behavior of solutions

$$\begin{aligned} & \underline{(z \longrightarrow -\infty) \quad Z \rightarrow 0,} \\ & F \sim Z^{1/2 \pm i\sqrt{\epsilon-1}}, \quad f \sim Z^{1 \pm i\sqrt{\epsilon-1}}, \quad \varphi \sim e^{\pm i\sqrt{\epsilon-1} z}; \\ & \underline{(z \longrightarrow +\infty) \quad Z \rightarrow +\infty,} \\ & F \sim e^{\pm Z}, \quad f = \sqrt{Z} e^{\pm Z}, \quad \varphi \sim e^{-z/2} \exp \left[\pm \sqrt{k_1^2 + k_2^2} e^z \right]. \end{aligned} \quad (1.7)$$

We now turn to the construction of exact solutions of (1.6) in the entire range of coordinate z . We seek solutions in the form of $f(Z) = Z^A e^{BZ} F(Z)$; equation (1.6) gives

$$\begin{aligned} & Z \frac{d^2 F}{dZ^2} + (2A - 1 + 2BZ) \frac{dF}{dZ} + \\ & + \left((B^2 - 1) Z - B(1 - 2A) + \frac{A(A - 2) + \epsilon}{Z} \right) F = 0. \end{aligned} \quad (1.8)$$

At A, B chosen according (for definiteness, we take the minus sign before the root in the expression for A ; assuming $\epsilon > 1$)

$$A = 1 - i\sqrt{\epsilon - 1}, \quad B^2 = 1, \quad (1.9)$$

the equation (1.8) is simplified

$$Z \frac{d^2 F}{dZ^2} + (2A - 1 + 2BZ) \frac{dF}{dZ} - B(1 - 2A) F = 0. \quad (1.10)$$

In (1.10), let us make another change $Z = y/2$:

$$y \frac{d^2 F}{dy^2} + (2A - 1 + By) \frac{dF}{dy} + B \left(A - \frac{1}{2}\right) F = 0, \quad (1.11)$$

with $B = -1$ it is an equation for the confluent hypergeometric function

$$\begin{aligned} y \frac{d^2 Y}{dZ^2} + (c - y) \frac{dY}{dy} - aY &= 0, \\ c = 2a, \quad a = A - 1/2 = 1/2 - i\sqrt{\epsilon - 1}, \\ f(Z) &= y^{a+1/2} e^{-y/2} Y(y). \end{aligned} \quad (1.12)$$

We use two pairs of linearly independent solutions[12]

$$Y_1 = \Phi(a, 2a, y), \quad Y_2 = y^{1-2a} \Phi(1-a, 2-2a, y)$$

and

$$Y_5 = \Psi(a, 2a, y), \quad Y_7 = e^y \Psi(a, 2a, -y), \quad (1.13)$$

These pairs of solutions are related by Kummer linear relations [12]

$$\begin{aligned} Y_5 &= \frac{\Gamma(1-2a)}{\Gamma(1-a)} Y_1 + \frac{\Gamma(2a-1)}{\Gamma(a)} Y_2, \\ Y_7 &= \frac{\Gamma(1-2a)}{\Gamma(1-a)} Y_1 - \frac{\Gamma(2a-1)}{\Gamma(a)} Y_2, \end{aligned} \quad (1.14a)$$

which after multiplication by $y^{a+1/2} e^{-y/2}$ take the form

$$\begin{aligned} f_5 &= \frac{\Gamma(1-2a)}{\Gamma(1-a)} f_1 + \frac{\Gamma(2a-1)}{\Gamma(a)} f_2, \\ f_7 &= \frac{\Gamma(1-2a)}{\Gamma(1-a)} f_1 - \frac{\Gamma(2a-1)}{\Gamma(a)} f_2. \end{aligned} \quad (1.14b)$$

Note that the solutions Y_1 and Y_2 describe the case of negative $z \rightarrow -\infty$ wave with the asymptotic behavior

$$\underline{z \rightarrow -\infty, (y \rightarrow 0)}$$

$$\begin{aligned} f_1 &\sim y^{a+1/2} = \left(2\sqrt{k_1^2 + k_2^2}\right)^{1-i\sqrt{\epsilon-1}} e^z e^{-i\sqrt{\epsilon-1}z}, \\ f_2 &\sim y^{a+1/2} y^{1-2a} = \left(2\sqrt{k_1^2 + k_2^2}\right)^{1+i\sqrt{\epsilon-1}} e^z e^{+i\sqrt{\epsilon-1}z}. \end{aligned} \quad (1.15)$$

Thus, for example, the function Y_5 (and the related φ_5) at negative $z \rightarrow -\infty$ behaves as a superposition of two plane waves according to

$$\begin{aligned} \varphi_5 \sim & \frac{\Gamma(1-2a)}{\Gamma(1-a)} \left(2\sqrt{k_1^2 + k_2^2} \right)^{1-i\sqrt{\epsilon-1}} e^{-i\sqrt{\epsilon-1}z} + \\ & + \frac{\Gamma(2a-1)}{\Gamma(a)} \left(2\sqrt{k_1^2 + k_2^2} \right)^{1+i\sqrt{\epsilon-1}} e^{+i\sqrt{\epsilon-1}z} . \end{aligned} \quad (1.16)$$

We define the reflection coefficient as the square modulus of the amplitude ratio in a superposition of plane waves

$$\begin{aligned} M_- e^{-i\sqrt{\epsilon-1}z} + M_+ e^{+i\sqrt{\epsilon-1}z} , \quad R = \left| \frac{M_-}{M_+} \right|^2 , \\ R = \left| \frac{\Gamma(1-2a)}{\Gamma(2a-1)} \frac{\Gamma(a)}{\Gamma(1-a)} \right|^2 . \end{aligned} \quad (1.17a)$$

We take into account

$$\begin{aligned} 1-2a &= +2i\sqrt{\epsilon-1} , \quad 2a-1 = -2i\sqrt{\epsilon-1} , \\ a &= 1/2 - i\sqrt{\epsilon-1} , \quad 1-a = 1/2 + i\sqrt{\epsilon-1} , \end{aligned}$$

then

$$R = \left| \frac{\Gamma(+2i\sqrt{\epsilon-1})}{\Gamma(-2i\sqrt{\epsilon-1})} \right|^2 \left| \frac{\Gamma(1/2 - i\sqrt{\epsilon-1})}{\Gamma(1/2 + i\sqrt{\epsilon-1})} \right|^2 \equiv 1 . \quad (1.17b)$$

We find the behavior of Y_5 at large y . Using the known asymptotic relation [12]

$$Y_5 = \Psi(a, c, y) \sim y^{-a} ,$$

we get

$$\begin{aligned} z \rightarrow +\infty , \quad f_5 &= y^{a+1/2} e^{-y/2} Y_5 \sim y^{1/2} e^{-y/2} \sim \\ &\sim \left(2\sqrt{k_1^2 + k_2^2} e^z \right)^{1/2} \exp \left(-\sqrt{k_1^2 + k_2^2} e^z \right) \rightarrow \exp^{-e^{+\infty}} = 0 . \end{aligned} \quad (1.18)$$

Thus, the solution f_5 describes the expected situation: wave going from the left is reflected with probability 1 on the effective barrier; behind the barrier the solutions sharply decrease to zero. It is easy to find the critical point, after which wave function must sharply decrease

$$\epsilon - 1 = (k_1^2 + k_2^2) e^{2z} \quad \Longrightarrow \quad z_0 = \ln \sqrt{\frac{\epsilon - 1}{k_1^2 + k_2^2}} , \quad (1.19a)$$

in the usual units, this critical point is described by the relation

$$z_0 = \rho \ln \sqrt{\frac{2mE\rho^2/\hbar^2 - 1}{(K_1^2 + K_2^2)\rho^2}} . \quad (1.19b)$$

The solution f_7 for $y \rightarrow +\infty$ goes to infinity

$$f_7 \sim y^{a+1/2} e^{-y/2} e^y y^{-a} = y^{+1/2} e^{+y/2} .$$

However, it is of no clear physical interpretation.

However, it is easy to interpret such a solution f_1 . Indeed, far on the right behind the barrier, taking into account the asymptotic formula [12]

$$y \rightarrow +\infty , \quad \Phi(A, C, y) = \frac{\Gamma(C)}{\Gamma(A)} e^y y^{A-C} ,$$

we get

$$z \rightarrow +\infty , \quad f_1 \sim y^{1/2} e^{y/2} \rightarrow +\infty^\infty ; \quad (1.20)$$

that is, for $z \rightarrow +\infty$ is a real function with infinite probability density. Far left is a plane wave propagating to the left.

Next we consider the analogue of this situation for particles with spin 1/2, described by the relativistic Dirac equation, when analysis is much more complicated.

2 The Dirac equation in the space H_3 , separation of variables

We start with the general covariant form of the Dirac equation [11]

$$\left[i \gamma^a \left(e_{(a)}^\alpha \frac{\partial}{\partial x^\alpha} + \frac{1}{2} \left(\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} \sqrt{-g} e_{(a)}^\alpha \right) \right) - m \right] \Psi(x) = 0 . \quad (2.1)$$

In the coordinate system (1.1) we use the tetrad

$$e_{(a)}^\beta = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & e^z & 0 & 0 \\ 0 & 0 & e^z & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} ; \quad (2.2)$$

eq. (2.1) takes the form

$$\left[\gamma^0 \frac{\partial}{\partial t} + \gamma^1 e^z \frac{\partial}{\partial x} + \gamma^2 e^z \frac{\partial}{\partial y} + \gamma^3 \left(\frac{\partial}{\partial z} - 1 \right) + im \right] \Psi = 0 . \quad (2.3)$$

Note that the addition of -1 about the operator ∂_z can be removed by substituting $\Psi = e^z \psi$. The following three operators $i\partial_t$, $i\partial_x$, $i\partial_y$ commute with the wave operator: solutions can be searched in the form

$$\Psi^{\epsilon, k_1, k_2} = e^{-i\epsilon t} e^{ik_1 x} e^{ik_2 y} \begin{pmatrix} f_1(z) \\ f_2(z) \\ f_3(z) \\ f_4(z) \end{pmatrix}. \quad (2.4)$$

Using the Dirac matrices in spinor basis, from (2.3) we find e equations for $f_i(z)$

$$\begin{aligned} -i\epsilon f_3 - ik_1 e^z f_4 - k_2 e^z f_4 - \left(\frac{d}{dz} - 1\right) f_3 + im f_1 &= 0, \\ -i\epsilon f_4 - ik_1 e^z f_3 + k_2 e^z f_3 + \left(\frac{d}{dz} - 1\right) f_4 + im f_2 &= 0, \\ -i\epsilon f_1 + ik_1 e^z f_2 + k_2 e^z f_2 + \left(\frac{d}{dz} - 1\right) f_1 + im f_3 &= 0, \\ -i\epsilon f_2 + ik_1 e^z f_1 - k_2 e^z f_1 - \left(\frac{d}{dz} - 1\right) f_2 + im f_4 &= 0. \end{aligned} \quad (2.5)$$

There is a generalized helicity operator which commutes with the operator of the wave equation:

$$\Sigma = \frac{1}{2} \left(e^z \gamma^2 \gamma^3 \frac{\partial}{\partial x} + e^z \gamma^3 \gamma^1 \frac{\partial}{\partial y} + \gamma^1 \gamma^2 \left(\frac{\partial}{\partial z} - 1 \right) \right). \quad (2.6)$$

Using the substitution (2.4) in the eigenvalues equation $\Sigma \Psi = p \Psi$ we obtain

$$\begin{aligned} k_1 e^z f_2 - ik_2 e^z f_2 - i \left(\frac{d}{dz} - 1 \right) f_1 &= p f_1, \\ k_1 e^z f_1 + ik_2 e^z f_1 + i \left(\frac{d}{dz} - 1 \right) f_2 &= p f_2, \\ k_1 e^z f_4 - ik_2 e^z f_4 - i \left(\frac{d}{dz} - 1 \right) f_3 &= p f_3, \\ k_1 e^z f_3 + ik_2 e^z f_3 + i \left(\frac{d}{dz} - 1 \right) f_4 &= p f_4. \end{aligned} \quad (2.7)$$

From equations (2.7) and (2.5), considered together, it follows a linear homogeneous system with respect to f_i

$$-i\epsilon f_3 - ip f_3 + im f_1 = 0,$$

$$\begin{aligned}
-i\epsilon f_4 - ip f_4 + im f_2 &= 0 , \\
-i\epsilon f_1 + ip f_1 + im f_3 &= 0 , \\
-i\epsilon f_2 + ip f_2 + im f_4 &= 0 .
\end{aligned} \tag{2.8}$$

We find two values for the p and the corresponding restrictions on the functions f_i :

$$p = \pm \sqrt{\epsilon^2 - m^2} , \quad f_3 = \frac{\epsilon - p}{m} f_1 , \quad f_4 = \frac{\epsilon - p}{m} f_2 . \tag{2.9}$$

Thus, we have three continuous quantum number ϵ , k_1 , k_2 and one discrete, which distinguishes the values $p = \pm \sqrt{\epsilon^2 - m^2}$. In view of (2.9), from four equations (2.5) we arrive at two equations for f_1 , f_2

$$\begin{aligned}
\left(\frac{d}{dz} - 1 - ip \right) f_1 + e^z (ik_1 + k_2) f_2 &= 0 , \\
\left(\frac{d}{dz} - 1 + ip \right) f_2 - e^z (ik_1 - k_2) f_1 &= 0 .
\end{aligned} \tag{2.10}$$

Note the symmetry of the equations with respect to change

$$f_1 \implies f_2 , \quad p \implies -p . \tag{2.11}$$

It is convenient to obtain solutions of similar equations in the flat space

$$\begin{aligned}
\left(\frac{d}{dz} - ip \right) f_1 + (ik_1 + k_2) f_2 &= 0 , \\
\left(\frac{d}{dz} + ip \right) f_2 - (ik_1 - k_2) f_1 &= 0 ,
\end{aligned} \tag{2.12}$$

so that

$$\begin{aligned}
f_2 &= -\frac{1}{ik_1 + k_2} \left(\frac{d}{dz} - ip \right) f_1 , \\
\left(\frac{d^2}{dz^2} + \epsilon^2 - m^2 - k_1^2 - k_2^2 \right) f_1 &= 0 .
\end{aligned} \tag{2.13}$$

Here, there exist two independent solutions (let $k_3 = +\sqrt{\epsilon^2 - m^2 - k_1^2 - k_2^2}$)

$$\begin{aligned}
f_1^{(1)} &= e^{+ik_3 z} , \quad f_2^{(1)} = -\frac{(+ik_3 - ip)}{ik_1 + k_2} e^{+ik_3 z} , \\
f_1^{(2)} &= e^{-ik_3 z} , \quad f_2^{(2)} = -\frac{(-ik_3 - ip)}{ik_1 + k_2} e^{-ik_3 z} .
\end{aligned} \tag{2.14}$$

The sign before k_3 determines the direction of the wave propagation, the sign of p defines the state of polarization. Generalized analogue of these solution is to be investigated in the hyperbolic space H_3 .

3 A special case of the waves along the z -axis

There exists a special case when $k_1 = 0$, $k_2 = 0$:

$$\Psi^{\epsilon,0,0}(t, z) = e^{-i\epsilon t} \begin{vmatrix} f_1(z) \\ f_2(z) \\ f_3(z) \\ f_4(z) \end{vmatrix}. \quad (3.1)$$

The equations change substantially (see (2.5))

$$\begin{aligned} -i\epsilon f_3 - \left(\frac{d}{dz} - 1\right) f_3 + im f_1 &= 0, \\ -i\epsilon f_4 + \left(\frac{d}{dz} - 1\right) f_4 + im f_2 &= 0, \\ -i\epsilon f_1 + \left(\frac{d}{dz} - 1\right) f_1 + im f_3 &= 0, \\ -i\epsilon f_2 - \left(\frac{d}{dz} - 1\right) f_2 + im f_4 &= 0, \end{aligned} \quad (3.2)$$

diagonalization of the operator Σ gives (see (2.7))

$$\begin{aligned} \left(\frac{d}{dz} - 1\right) f_1 &= ip f_1, & \left(\frac{d}{dz} - 1\right) f_2 &= -ip f_2, \\ \left(\frac{d}{dz} - 1\right) f_3 &= ip f_3, & \left(\frac{d}{dz} - 1\right) f_4 &= -ip f_4. \end{aligned} \quad (3.3)$$

Considering equations (3.2) and (3.3) together, we arrive at the linear system

$$\begin{aligned} (\epsilon + p) f_3 - m f_1 &= 0, & (\epsilon + p) f_4 - m f_2 &= 0, \\ (\epsilon - p) f_1 - m f_3 &= 0, & (\epsilon - p) f_2 - m f_4 &= 0. \end{aligned} \quad (3.4)$$

From this it follows

$$\begin{aligned} (\epsilon^2 - p^2)^2 - m^4 &= 0 \quad \implies \quad p = \pm \sqrt{\epsilon^2 - m^2}, \\ f_4 = \frac{\epsilon - p}{m} f_2 = \frac{m}{\epsilon + p} f_2, & \quad f_3 = \frac{\epsilon - p}{m} f_1 = \frac{m}{\epsilon + p} f_1. \end{aligned} \quad (3.5)$$

Equation (3.2) reduce to

$$\left(\frac{\partial}{\partial z} - 1 - ip\right) f_1 = 0 \quad \implies \quad f_1 = C_1 e^z e^{+ipz},$$

$$\left(\frac{\partial}{\partial z} - 1 + ip \right) f_2 = 0 \quad \implies \quad f_2 = C_2 e^z e^{-ipz} . \quad (3.6)$$

Solutions more simple to interpret are

$$C_1 = 1 , \quad C_2 = 0 , \quad \Psi^{\epsilon 00p}(t, z) = \begin{vmatrix} 1 \\ 0 \\ \frac{\epsilon - p}{m} \\ 0 \end{vmatrix} e^z e^{+ipz}; \quad (3.7a)$$

$$C_1 = 0 , \quad C_2 = 1 , \quad \Psi^{\epsilon 00p}(t, z) = \begin{vmatrix} 0 \\ 1 \\ 0 \\ \frac{\epsilon - p}{m} \end{vmatrix} e^z e^{-ipz}. \quad (3.7b)$$

Obviously, the factor e^z in the solutions will be compensated when considering any bilinear structure of the wave functions (with their subsequent multiplication by $\sqrt{-g} dx dy dz$).

4 Construction of solutions in the general case

Let us turn to (2.10) and introduce a new variable Z :

$$\begin{aligned} \sqrt{k_1^2 + k_2^2} e^z &= Z , \quad Z \in (0, +\infty) , \\ \left(Z \frac{d}{dZ} - 1 - ip \right) f_1 + Z \sqrt{\frac{k_2 + ik_1}{k_2 - ik_1}} f_2 &= 0 , \\ \left(Z \frac{d}{dZ} - 1 + ip \right) f_2 + Z \sqrt{\frac{k_2 - ik_1}{k_2 + ik_1}} f_1 &= 0 . \end{aligned}$$

It is convenient to define a parameter

$$e^{ia} = \sqrt{\frac{k_2 + ik_1}{k_2 - ik_1}} ,$$

then

$$\begin{aligned} \left(Z \frac{d}{dZ} - 1 - ip \right) f_1 + Z e^{+ia} f_2 &= 0 , \\ \left(Z \frac{d}{dZ} - 1 + ip \right) f_2 + Z e^{-ia} f_1 &= 0 . \end{aligned} \quad (4.1a)$$

Note that the factor $e^{i\alpha}$ can be removed by changing the notation for f_2

$$e^{+ia} f_2 \implies f_2 ;$$

then the system takes the form

$$\begin{aligned} \left(Z \frac{d}{dZ} - 1 - ip \right) f_1 + Z f_2 &= 0, \\ \left(Z \frac{d}{dZ} - 1 + ip \right) f_2 + Z f_1 &= 0. \end{aligned} \quad (4.1b)$$

From (4.1a) we get two second order differential equations for f_1 and f_2 :

$$Z \frac{d^2 f_1}{dZ^2} - 2 \frac{d f_1}{dZ} + \left(\frac{p^2 + ip + 2}{Z} - Z \right) f_1 = 0, \quad (4.2a)$$

$$Z \frac{d^2 f_2}{dZ^2} - 2 \frac{d f_2}{dZ} + \left(\frac{p^2 - ip + 2}{Z} - Z \right) f_2 = 0. \quad (4.2b)$$

Note the symmetry between the equations: they are transformed into each other when changing $p \rightarrow -p$. Furthermore, it should be noted that unlike the case of flat space (see (2.13)), here second-order equations for the functions f_1 and f_2 depend explicitly on the first degree of p , that is depends on the state of polarization of spinor waves.

Considering eq. (4.2a), let us use a substitution $f_1(Z) = Z^A e^{BZ} F_1(Z)$:

$$\begin{aligned} & Z \frac{d^2 F_1}{dZ^2} + (2A - 2 + 2BZ) \frac{d F_1}{dZ} + \\ & + \left[(B^2 - 1)Z + 2B(A - 1) + \frac{(A - ip - 1)(A + ip - 2)}{Z} \right] F_1 = 0. \end{aligned} \quad (4.3)$$

At A and B chosen according

$$A = +ip + 1, \quad -ip + 2, \quad B = \pm 1, \quad (4.4)$$

(4.3) becomes simpler

$$Z \frac{d^2 F_1}{dZ^2} + (2A - 2 + 2BZ) \frac{d F_1}{dZ} + 2B(A - 1)F_1 = 0. \quad (4.5)$$

The resulting equation with another change of $Z = y/2$ will transform into

$$y \frac{d^2 F_1}{dy^2} + (2A - 2 + By) \frac{d F_1}{dy} + B(A - 1)F_1 = 0. \quad (4.6a)$$

When $B = -1$, it coincides with an equation for the confluent hypergeometric function with parameters (for definiteness let it be $A = +ip + 1$)

$$y \frac{d^2 \Phi}{dy^2} + (c - y) \frac{d \Phi}{dy} - a \Phi = 0, \quad a = +ip, \quad c = 2a = +2ip. \quad (4.6b)$$

Two linearly independent solutions are [12]

$$\begin{aligned} F_1^{(1)}(y) &= \Phi(a, c, y) , \\ F_1^{(2)}(y) &= y^{1-c} \Phi(a - c + 1, 2 - c, y) . \end{aligned} \quad (4.6c)$$

Consider the equation (4.2b). Using the above-noted symmetry, we obtain

$$\begin{aligned} f_2 &= y^{a'+1} e^{-y/2} F_2(y) , \quad a' = -ip , \quad c' = 2a' = -2ip , \\ F_2^{(1)} &= \Phi(a', c', y) , \\ F_2^{(2)} &= y^{1-c'} \Phi(a' - c' + 1, 2 - c', y) . \end{aligned} \quad (4.7)$$

It is convenient to employ one independent parameter a :

$$\begin{aligned} f_1 &= y^{a+1} e^{-y/2} F_1(y) , \\ F_1^{(1)}(y) &= \Phi(a, 2a, y) , \\ F_1^{(2)}(y) &= y^{1-2a} \Phi(1 - a, 2 - 2a, y) ; \end{aligned} \quad (4.8a)$$

$$\begin{aligned} f_2 &= y^{-a+1} e^{-y/2} F_2(y) , \\ F_2^{(1)}(y) &= \Phi(-a, -2a, y) , \\ F_2^{(2)}(y) &= y^{1+2a} \Phi(1 + a, 2 + 2a, y) . \end{aligned} \quad (4.8b)$$

The functions f_1, f_2 (note that before now we did not find possible numerical factors at them) must be related by the first-order operators (see (4.1))

$$\begin{aligned} \left(y \frac{d}{dy} - 1 - a \right) f_1 - \frac{y}{2} e^{+ia} f_2 &= 0 , \\ \left(y \frac{d}{dy} - 1 + a \right) f_2 - \frac{y}{2} e^{-ia} f_1 &= 0 ; \end{aligned}$$

these relationships can be translated to the functions F_1, F_2 , which results in

$$\frac{dF_1}{dy} - \frac{1}{2} (F_1 + y^{-2a} e^{+ia} F_2) = 0 , \quad (4.9a)$$

$$\frac{dF_2}{dy} - \frac{1}{2} (F_2 + y^{+2a} e^{-ia} F_1) = 0 . \quad (4.9b)$$

These equations relate functions in the following pairs

$$F_1^{(1)}(y) \text{ --- } F_2^{(2)}(y) , \quad F_1^{(2)}(y) \text{ --- } F_2^{(1)}(y) .$$

For each pair one should find the relative coefficient of the two functions.

Let us substitute expressions for $F_1^{(1)}(y)$ and $F_2^{(2)}(y)$ with some numerical coefficients

$$F_1^{(1)}(y) = r_1^{(1)} \Phi(a, 2a, y) , \quad F_2^{(2)}(y) = r_2^{(2)} y^{1+2a} \Phi(a+1, 2+2a, y)$$

in eq. (4.9a). Performing the necessary differentiation

$$r_1^{(1)} [\Phi(a+1, 2a+1, y) - \Phi(a, 2a, y)] - r_2^{(2)} e^{+ia} y \Phi(a+1, 2a+2, y) = 0 ,$$

and transformation using the relations for contiguous functions, we find the relative factor

$$r_1^{(1)} = 2 e^{+ia} (2a+1) r_2^{(2)} . \quad (4.10)$$

The same result can be obtained by substituting the expression for $F_1^{(1)}(y)$ and $F_2^{(2)}(y)$ in the equation (4.9b). The resulting ratio can also be obtained by using the expansions of solutions near $y = 0$

$$F_1^{(1)} = r_1^{(1)} \left(1 + \frac{1}{2}y + \frac{(a+1)}{2(2a+1)}y^2 + \dots \right), \quad F_2^{(2)}(y) = r_2^{(2)} y^{1+2a} ;$$

the equation

$$\frac{dF_1^{(1)}}{dy} - \frac{1}{2}F_1^{(1)} - \frac{1}{2}y^{-2a} e^{+ia} F_2^{(2)} = 0$$

gives

$$r_1^{(1)} \left(\frac{1}{2} + \frac{a+1}{2a+1}y - \frac{1}{2} - \frac{1}{2}y \right) - e^{+ia} r_2^{(2)} y = 0 \implies$$

$$r_1^{(1)} \frac{1}{2(2a+1)} - e^{+ia} r_2^{(2)} = 0 .$$

Now let us substitute expressions for $F_1^{(2)}(y)$ and $F_2^{(1)}(y)$ of (4.8)

$$F_1^{(2)}(y) = r_1^{(2)} y^{1-2a} \Phi(1-a, 2-2a, y) ,$$

$$F_2^{(1)}(y) = r_2^{(1)}(y) \Phi(-a, -2a, y) \quad (4.11a)$$

in the equation (4.9b). Performing the differentiation

$$r_1^{(2)} [\Phi(1-a, 1-2a, y) - \Phi(-a, -2a, y)] -$$

$$-r_2^{(1)} e^{-ia} y \Phi(1-a, 2-2a, y) = 0$$

and the corresponding transformations, we arrive at

$$r_1^{(2)} = 2 e^{-ia} (1-2a) r_2^{(1)} . \quad (4.11b)$$

Thus, the construction of two linearly independent solutions of (4.1):

$$\begin{aligned} f_1 &= r_1^{(1)} e^{-y/2} y^{a+1} \Phi(a, 2a, y) , \\ f_2 &= r_2^{(2)} e^{-y/2} y^{2+a} \Phi(a+1, 2+2a, y) , \\ r_1^{(1)} &= 2 e^{+ia} (1+2a) r_2^{(2)} ; \end{aligned} \quad (4.12a)$$

$$\begin{aligned} f_2 &= r_2^{(1)} e^{-y/2} y^{-a+1} \Phi(-a, -2a, y) , \\ f_1 &= r_1^{(2)} e^{-y/2} y^{2-a} \Phi(1-a, 2-2a, y) , \\ r_1^{(2)} &= 2 e^{-ia} (1-2a) r_2^{(1)} . \end{aligned} \quad (4.12b)$$

We simplify the formulas by setting $r_2^{(1)} = 1$, $r_2^{(2)} = 1$; the result is

$$\begin{aligned} I \quad f_1 &= M_+ e^{-y/2} y^{1+a} \Phi(a, 2a, y) , \\ f_2 &= e^{-y/2} y^{2+a} \Phi(a+1, 2+2a, y) , \\ M_+ &= [2 e^{+ia} (1+2a)] ; \end{aligned} \quad (4.13a)$$

$$\begin{aligned} II \quad f_1 &= M_- e^{-y/2} y^{2-a} \Phi(1-a, 2-2a, y) , \\ f_2 &= e^{-y/2} y^{1-a} \Phi(-a, -2a, y) , \\ M_- &= [2 e^{-ia} (1-2a)] . \end{aligned} \quad (4.13b)$$

Remind that $a = ip = \pm i\sqrt{\epsilon^2 - m^2}$; the sign of p is associated with the polarization state of the spinor waves; types I and II are supposed to be associated with the directions of wave propagation: to the left or to the right.

Let us consider asymptotic properties of the solutions. The coordinate $y = 2\sqrt{k_1^2 + k_2^2} e^z$ tends to (see (1.2))

$$\begin{aligned} q \longrightarrow -1 , \quad y &= 2\sqrt{k_1^2 + k_2^2} e^z \longrightarrow 0 , \\ q \longrightarrow +1 , \quad y &= 2\sqrt{k_1^2 + k_2^2} e^z \longrightarrow +\infty . \end{aligned} \quad (4.13c)$$

The behavior of solutions in boundary point $y \rightarrow 0$ ($z \rightarrow -\infty$):

I

$$\begin{aligned} f_1 &= M_+ y^{1+a} = M_+ \left(2\sqrt{k_1^2 + k_2^2} e^z \right)^{1+ip}, \\ f_2 &= y^{2+a} = \left(2\sqrt{k_1^2 + k_2^2} e^z \right)^{2+ip}; \end{aligned} \quad (4.14a)$$

II

$$\begin{aligned} f_1 &= M_- y^{2-a} = \left(2\sqrt{k_1^2 + k_2^2} e^z \right)^{2-ip}, \\ f_2 &= y^{1-a} = \left(2\sqrt{k_1^2 + k_2^2} e^z \right)^{1-ip}. \end{aligned} \quad (4.14b)$$

For large values of y one should use the asymptotic formula [12]

$$y \rightarrow +\infty, \quad \Phi(A, C, y) = \frac{\Gamma(C)}{\Gamma(A)} e^y y^{A-C}.$$

So we get (for $\underline{z \rightarrow +\infty, y \rightarrow +\infty}$)

$$I \quad f_1 = M_+ e^{y/2} y \frac{\Gamma(2a)}{\Gamma(a)}, \quad f_2 = e^{y/2} y \frac{\Gamma(2+2a)}{\Gamma(a+1)}, \quad (4.15a)$$

$$II \quad f_1 = M_- e^{y/2} y \frac{\Gamma(2-2a)}{\Gamma(1-a)}, \quad f_2 = e^{y/2} y \frac{\Gamma(-2a)}{\Gamma(-a)}. \quad (4.15b)$$

To conclude this section we consider the limiting process in the constructed solutions (4.13a), (4.13b) to the case of the flat space. This will allow a better understanding of the obtained results in the Lobachevsky space.

To this end, we first need to go to the usual dimensional quantities:

$$z = \frac{z_3}{R}, \quad m = \frac{McR}{\hbar}, \quad \epsilon = \frac{ER}{c\hbar},$$

$$p = +\sqrt{\epsilon^2 - m^2} = +R \sqrt{E^2/c^2\hbar^2 - M^2c^2/\hbar^2} = Rp_0,$$

$$k_1 = \frac{P_1 R}{c\hbar}, \quad k_2 = \frac{P_2 R}{c\hbar}, \quad \sqrt{k_1^2 + k_2^2} = R \frac{\sqrt{P_1^2 + P_2^2}}{c\hbar} = RK_\perp,$$

$$a = ip = iRp_0, \quad c = 2a = i2Rp_0,$$

$$y = 2\sqrt{k_1^2 + k_2^2}e^z = 2RK_\perp(1 + \frac{x_3}{R} + \dots) \longrightarrow 2RK_\perp. \quad (4.16)$$

Let us consider the solutions (4.13a)

$$\begin{aligned} I \quad f_1 &= M_+ e^{-y/2} y^{1+a} \Phi(a, 2a, y), \\ f_2 &= e^{-y/2} y^{2+a} \Phi(a+1, 2+2a, y), \\ M_+ &= [2 e^{+ia} (1+2a)] ; \end{aligned}$$

taking into account

$$\begin{aligned} \frac{a}{c}y &= \frac{1}{2}y \implies RK_\perp, \\ \frac{1}{2!} \frac{a(a+1)}{c(c+1)} y^2 &= \frac{1}{2!} \frac{1/2(1/2+1/c)}{(1+1/c)} y^2 \implies \frac{1}{2!} (RK_\perp)^2, \\ \frac{1}{3!} \frac{a(a+1)(a+2)}{c(c+1)(c+2)} y^3 &= \frac{1}{3!} \frac{1/2(1/2+1/c)(1/2+2/c)}{(1+1/c)(1+2/c)} y^3 \implies \\ &\quad \frac{1}{3!} (RK_\perp)^3 \dots, \end{aligned} \quad (4.17)$$

we get

$$\begin{aligned} e^{-y/2} &\implies e^{-RK_\perp}, \quad \Phi(a, 2a, y) \implies e^{RK_\perp}, \\ e^{-y/2} &\implies e^{-RK_\perp}, \quad \Phi(a+1, 2a+2, y) \implies e^{RK_\perp}, \end{aligned}$$

and further

$$\begin{aligned} I \quad f_1 &\implies M_+ (2RK_\perp e^z)^{1+iRp_0} \sim e^{ix_3 p_0}, \\ f_2 &\implies (2RK_\perp e^z)^{2+iRp_0} \sim e^{ix_3 p_0}. \end{aligned} \quad (4.18)$$

Similarly, we find

$$\begin{aligned} II \quad f_2 &= e^{-y/2} y^{1-a} \Phi(-a, -2a, y) \sim e^{-ix_3 p_0}, \\ f_1 &= M_- e^{-y/2} y^{2-a} \Phi(1-a, 2-2a, y) \sim e^{-ix_3 p_0}. \end{aligned} \quad (4.19)$$

We may conclude that solutions of the type I (in curved model H_3) provide us with extension for the flat waves in Minkowski space of the type e^{+ikz} ; whereas solutions of the type II represent extension for the flat waves in Minkowski space of the type e^{-ikz} .

5 The case of the Weyl neutrino

Let us restrict the above analysis to the case of 2-component Weyl neutrino. It is enough to specify the Dirac equation in the spinor basis (see [11])

$$\begin{aligned} i \sigma^\alpha(x) [\partial_\alpha + \Sigma_\alpha(x)] \xi(x) &= m \eta(x) , \\ i \bar{\sigma}^\alpha(x) [\partial_\alpha + \bar{\Sigma}_\alpha(x)] \eta(x) &= m \xi(x) \end{aligned}$$

and impose relation $m = 0$, thus we will obtain the equations of the Weyl neutrino (wave function $\eta(x)$) and antineutrinos (with the wave function $\xi(x)$).

Substitution for η is

$$\eta^{\epsilon, k_1, k_2} = e^{-i\epsilon t} e^{ik_1 x} e^{ik_2 y} \begin{vmatrix} f_3(z) \\ f_4(z) \end{vmatrix} . \quad (5.1)$$

Note that neutrino wave function has only three quantum numbers ϵ, k_1, k_2 . The system of equations after separation of variables has the form

$$\begin{aligned} -i\epsilon f_3 - ik_1 e^z f_4 - k_2 e^z f_4 - \left(\frac{\partial}{\partial z} - 1\right) f_3 &= 0 , \\ -i\epsilon f_4 - ik_1 e^z f_3 + k_2 e^z f_3 + \left(\frac{\partial}{\partial z} - 1\right) f_4 &= 0 \end{aligned} \quad (5.2)$$

or (replacing f_3 on h_1 and f_4 on h_2)

$$\begin{aligned} \left(\frac{d}{dz} - 1 + i\epsilon\right) h_1 + e^z (ik_1 + k_2) h_2 &= 0 , \\ \left(\frac{d}{dz} - 1 - i\epsilon\right) h_2 - e^z (ik_1 - k_2) h_1 &= 0 . \end{aligned} \quad (5.3)$$

One can use the solution obtained above (for the system (2.10)), replacing everywhere p on $-\epsilon$. Thus, we get two linearly independent solutions of the system (5.3)

$$\begin{aligned} h_1 &= r_1^{(1)} e^{-y/2} y^{a+1} \Phi(a, 2a, y) , \\ h_2 &= r_2^{(2)} e^{-y/2} y^{2+a} \Phi(a+1, 2+2a, y) , \\ r_1^{(1)} &= 2 e^{+ia} (1+2a) r_2^{(2)} ; \end{aligned} \quad (5.4a)$$

$$\begin{aligned} h_2 &= r_2^{(1)} e^{-y/2} y^{-a+1} \Phi(-a, -2a, y) , \\ h_1 &= r_1^{(2)} e^{-y/2} y^{2-a} \Phi(1-a, 2-2a, y) , \\ r_1^{(2)} &= 2 e^{-ia} (1-2a) r_2^{(1)} , \end{aligned} \quad (5.4b)$$

where

$$a = -i\epsilon , \quad c = 2a = -2i\epsilon . \quad (5.4c)$$

6 On the non-relativistic Pauli approximation

We will carry out a procedure of the non-relativistic approximation directly in the separated equations for relativistic Dirac case. To this end, we turn to equations (2.5)

$$\begin{aligned}
-i\epsilon f_2 + ik_1 e^z f_1 - k_2 e^z f_1 - \left(\frac{d}{dz} - 1\right) f_2 + im f_4 &= 0, \\
-i\epsilon f_4 - ik_1 e^z f_3 + k_2 e^z f_3 + \left(\frac{d}{dz} - 1\right) f_4 + im f_2 &= 0, \\
-i\epsilon f_3 - ik_1 e^z f_4 - k_2 e^z f_4 - \left(\frac{d}{dz} - 1\right) f_3 + im f_1 &= 0, \\
-i\epsilon f_1 + ik_1 e^z f_2 + k_2 e^z f_2 + \left(\frac{d}{dz} - 1\right) f_1 + im f_3 &= 0.
\end{aligned} \tag{6.1}$$

We introduce new functions

$$\begin{aligned}
\frac{f_1 + f_3}{2} &= f, & \frac{f_1 - f_3}{2i} &= g, \\
\frac{f_2 + f_4}{2} &= F, & \frac{f_2 - f_4}{2i} &= G.
\end{aligned} \tag{6.2}$$

From (6.1), one obtains equations for f, F, g, G :

$$\begin{aligned}
\left(\frac{d}{dz} - 1\right) G - e^z (ik_1 - k_2) g + (\epsilon - m) F &= 0, \\
\left(\frac{d}{dz} - 1\right) F - e^z (ik_1 - k_2) f - (\epsilon + m) G &= 0, \\
\left(\frac{d}{dz} - 1\right) g + e^z (ik_1 + k_2) G - (\epsilon - m) f &= 0, \\
\left(\frac{d}{dz} - 1\right) f + e^z (ik_1 + k_2) F + (\epsilon + m) g &= 0.
\end{aligned} \tag{6.3}$$

Now we should make a formal change $\epsilon \implies m + E$ (this is equivalent to separating the rest energy by a factor e^{-imt}); as a result we get

$$\begin{aligned}
\left(\frac{d}{dz} - 1\right) G - e^z (ik_1 - k_2) g + E F &= 0, \\
\left(\frac{d}{dz} - 1\right) F - e^z (ik_1 - k_2) f - (E + 2m) G &= 0,
\end{aligned}$$

$$\begin{aligned}
& \left(\frac{d}{dz} - 1\right) g + e^z (ik_1 + k_2) G - E f = 0, \\
& \left(\frac{d}{dz} - 1\right) f + e^z (ik_1 + k_2) F + (E + 2m) g = 0.
\end{aligned} \tag{6.4}$$

The condition for the applicability of the nonrelativistic approximation is the following relation $E + 2m \approx 2m$; which results in

$$\begin{aligned}
& \left(\frac{d}{dz} - 1\right) G - e^z (ik_1 - k_2) g + E F = 0, \\
& G = \frac{1}{2m} \left[\left(\frac{d}{dz} - 1\right) F - e^z (ik_1 - k_2) f \right], \\
& \left(\frac{d}{dz} - 1\right) g + e^z (ik_1 + k_2) G - E f = 0, \\
& g = -\frac{1}{2m} \left[\left(\frac{d}{dz} - 1\right) f + e^z (ik_1 + k_2) F \right].
\end{aligned} \tag{6.5}$$

Excluding two small components of g , G , we arrive at two equations for the big components f and F :

$$\begin{aligned}
& \left[\frac{d^2}{dz^2} - 2 \frac{d}{dz} + 1 - e^{2z} (k_1^2 + k_2^2) + 2mE \right] F - e^z (ik_1 - k_2) f = 0, \\
& \left[\frac{d^2}{dz^2} - 2 \frac{d}{dz} + 1 - e^{2z} (k_1^2 + k_2^2) + 2mE \right] f + e^z (ik_1 + k_2) F = 0.
\end{aligned} \tag{6.6}$$

We remind that (see (2.9))

$$\begin{aligned}
f &= \frac{f_1(z) + f_3(z)}{2} = \frac{1+A}{2} f_1(z), \\
F &= \frac{f_2(z) + f_4(z)}{2} = \frac{1+A}{2} f_2(z), \\
A &= \frac{\epsilon \pm p}{m}, \quad p = \pm \sqrt{\epsilon^2 - m^2} = \\
&= \pm \sqrt{(E+m)^2 - m^2} \approx \pm \sqrt{2mE};
\end{aligned} \tag{6.7a}$$

sign \pm correspond to two different polarizations of the non-relativistic electron. Therefore, equation (6.6) can be represented as follows (remember that now the parameter p defined by the nonrelativistic expression (6.7a)):

$$\begin{aligned} \left[\frac{d^2}{dz^2} - 2 \frac{d}{dz} + 1 - e^{2z} (k_1^2 + k_2^2) + 2mE \right] f_2 - e^z (ik_1 - k_2) f_1 &= 0, \\ \left[\frac{d^2}{dz^2} - 2 \frac{d}{dz} + 1 - e^{2z} (k_1^2 + k_2^2) + 2mE \right] f_1 + e^z (ik_1 + k_2) f_2 &= 0. \end{aligned} \quad (6.7b)$$

These equations can be written as (for definiteness, let $p = +\sqrt{2mE}$)

$$\begin{aligned} \left[\left(\frac{d}{dz} - 1 - ip \right) \left(\frac{d}{dz} - 1 + ip \right) - e^{2z} (k_1^2 + k_2^2) \right] f_2 - e^z (ik_1 - k_2) f_1 &= 0, \\ \left[\left(\frac{d}{dz} - 1 - ip \right) \left(\frac{d}{dz} - 1 + ip \right) - e^{2z} (k_1^2 + k_2^2) \right] f_1 + e^z (ik_1 + k_2) f_2 &= 0. \end{aligned} \quad (6.8)$$

We take into account (2.10)

$$\begin{aligned} \left(\frac{d}{dz} - 1 - ip \right) f_1 + e^z (ik_1 + k_2) f_2 &= 0, \\ \left(\frac{d}{dz} - 1 + ip \right) f_2 - e^z (ik_1 - k_2) f_1 &= 0. \end{aligned}$$

Thus we arrive at equations

$$\begin{aligned} &\left(\frac{d}{dz} - 1 - ip \right) (e^z (ik_1 - k_2) f_1) - \\ &- e^z (ik_1 - k_2) \left(\frac{d}{dz} - 1 - ip \right) f_1 - e^z (ik_1 - k_2) f_1 = 0, \\ &\left(\frac{d}{dz} - 1 + ip \right) (-e^z (ik_1 + k_2) f_2) + \\ &+ e^z (ik_1 + k_2) \left(\frac{d}{dz} - 1 + ip \right) f_2 + e^z (ik_1 + k_2) f_2 = 0. \end{aligned} \quad (6.9)$$

It is easy to see that these two identities of the form $0 = 0$.

Thus, the solutions found above for the relativistic Dirac equation in the approximation $p \approx \sqrt{2mE}$ are solutions of two-component Pauli equation.

7 On representation of solutions in terms of Bessel functions

Returning to the basic system (4.1) in the form

$$\begin{aligned} \left(Z \frac{d}{dZ} - 1 - ip \right) \sqrt{k_2 - ik_1} f_1 + Z \sqrt{k_2 + ik_1} f_2 &= 0, \\ \left(Z \frac{d}{dZ} - 1 + ip \right) \sqrt{k_2 + ik_1} f_2 + Z \sqrt{k_2 - ik_1} f_1 &= 0, \end{aligned}$$

let us express it in terms of new functions

$$\sqrt{k_2 - ik_1} f_1 = e^z \varphi_1, \quad \sqrt{k_2 + ik_1} f_2 = e^z \varphi_2; \quad (7.1)$$

as a result we obtain

$$\begin{aligned} \left(Z \frac{d}{dZ} - ip \right) \varphi_1 + Z \varphi_2 &= 0, \\ \left(Z \frac{d}{dZ} + ip \right) \varphi_2 + Z \varphi_1 &= 0. \end{aligned} \quad (7.2)$$

Let us translate them to the new variable $x = iZ = i\sqrt{k_1^2 + k_2^2}e^z$:

$$\begin{aligned} \left(x \frac{d}{dx} - ip \right) \varphi_1 - ix \varphi_2 &= 0, \\ \left(x \frac{d}{dx} + ip \right) \varphi_2 - ix \varphi_1 &= 0. \end{aligned} \quad (7.3)$$

From (7.3) it follow two second-order equations

$$\begin{aligned} \left(\frac{d^2}{dx^2} + 1 + \frac{p^2 + ip}{x^2} \right) \varphi_1 &= 0, \\ \left(\frac{d^2}{dx^2} + 1 + \frac{p^2 - ip}{x^2} \right) \varphi_2 &= 0. \end{aligned} \quad (7.4)$$

Separating the factor \sqrt{x} : $\varphi_1 = \sqrt{x}F_1$, $\varphi_2 = \sqrt{x}F_2$, we arrive at the two Bessel equations

$$\begin{aligned} \left(\frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} + 1 + \frac{p^2 + ip - 1/4}{x^2} \right) F_1 &= 0, \\ \left(\frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} + 1 + \frac{p^2 - ip - 1/4}{x^2} \right) F_2 &= 0. \end{aligned} \quad (7.5)$$

Since we are (especially) interested in the case of the Weyl neutrino (with helicity -1), we will continue to consider in detail the case of negative $p = -\sqrt{\epsilon^2 - m^2}$. To avoid new notation, we will make a minus sign in front of p in (7.5), so that

$$\begin{aligned} \left(\frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} + 1 + \frac{p^2 - ip - 1/4}{x^2} \right) F_1 &= 0, \\ \left(\frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} + 1 + \frac{p^2 + ip - 1/4}{x^2} \right) F_2 &= 0, \end{aligned} \quad (7.6)$$

where $p = +\sqrt{\epsilon^2 - m^2} > 0$.

Thus, for polarization states with $p = +\sqrt{\epsilon^2 - m^2} > 0$, the functions $F_1(y), F_2(y)$ satisfy the Bessel equations [13]

$$\begin{aligned} \left(\frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} + 1 - \frac{(ip + 1/2)^2}{x^2} \right) F_1 &= 0, \\ \nu = -ip - 1/2, \quad F_1 &= J_{+\nu}(x), \quad J_{-\nu}(x). \end{aligned} \quad (7.7a)$$

$$\begin{aligned} \left(\frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} + 1 - \frac{(-ip + 1/2)^2}{x^2} \right) F_2 &= 0, \\ \mu = -ip + 1/2 = \nu + 1, \quad F_2 &= J_{+\mu}(x), \quad J_{-\mu}(x). \end{aligned} \quad (7.7b)$$

Transition to polarization states with $p' = -\sqrt{\epsilon^2 - m^2} > 0$ is achieved by replacement in (7.7) positive p to negative $p' = -p$, which results in

$$\nu' = +ip - 1/2 = -\mu, \quad \mu' = +ip + 1/2 = \nu' + 1 = -\nu. \quad (7.7c)$$

Since we have to follow explicit form of both functions F_1, F_2 (up to a relative factor), let us return to the first order equations (now in the variable $x = iZ$); with notation $\nu = -ip - 1/2$ it reads

$$\left(x \frac{d}{dx} - \nu \right) F_1 = -ix F_2, \quad (7.8a)$$

$$\left(x \frac{d}{dx} + \nu + 1 \right) F_2 = -ix F_1. \quad (7.8b)$$

From the above it is known that functions $F_1(x), F_2(x)$ satisfy the Bessel equations (7.6). Let us recall the well-known recurrence formulas for the solutions of Bessel equation [13] – write them in a convenient form

$$\left(x \frac{d}{dx} - \nu \right) F_\nu(x) = -x F_{\nu+1}(x), \quad (7.9a)$$

$$\left(x \frac{d}{dx} - \nu\right) F_{-\nu}(x) = +x F_{-\nu-1}(x) ; \quad (7.9b)$$

here by $F_{\pm\nu}$ can be understood either of the Bessel functions: $J_{\pm\nu}$, or Hankel functions $H_{\pm\nu}^1, H_{\pm\nu}^2$, or Neumann functions $N_{\pm\nu}(x)$.

Comparing eq. (7.9a) with eq. of (7.8a), we find two types of solutions:

in Bessel's functions

$$\begin{aligned} I \quad F_1^I(x) &= J_{+\nu}(x) , & F_2^I(x) &= -i J_{+(\nu+1)}(x) ; \\ II \quad F_1^{II}(x) &= J_{-\nu}(x) , & F_2^{II}(x) &= +i J_{-(\nu+1)}(x) ; \end{aligned} \quad (7.10)$$

in Hankel's functions

$$\begin{aligned} I \quad F_1^I(x) &= H_{+\nu}^1(x) , & F_2^I(x) &= -i H_{+(\nu+1)}^1(x) ; \\ II \quad F_1^{II}(x) &= H_{+\nu}^2(x) , & F_2^{II}(x) &= -i H_{+(\nu+1)}^2(x) ; \end{aligned} \quad (7.11a)$$

$$\begin{aligned} I' \quad F_1(x) &= H_{-\nu}^1(x) , & F_2(x) &= +i H_{-(\nu+1)}^1(x) ; \\ II' \quad F_1(x) &= H_{-\nu}^2(x) , & F_2(x) &= +i H_{-(\nu+1)}^2(x) ; \end{aligned} \quad (7.11b)$$

note that $H_{-\nu}^1(x) = e^{i\nu\pi} H_{\nu}^2(x)$, so the primed cases I', II' coincide respectively with II, I and by this reason will not be considered below.

And in Neumann functions

$$\begin{aligned} I \quad F_1^I(x) &= N_{+\nu}(x) , & F_2^I(x) &= -i N_{+(\nu+1)}(x) ; \\ II \quad F_1^{II}(x) &= N_{-\nu}(x) , & F_2^{II}(x) &= +i N_{-(\nu+1)}(x) . \end{aligned} \quad (7.12)$$

First, let us detail solutions in Bessel's functions [13].

In the region $z \rightarrow -\infty$, $x \rightarrow i0$,

$$\begin{aligned} I \quad F_1^I(x) &= J_{+\nu}(x) = \left(\frac{x}{2}\right)^{\nu} = (i\lambda)^{\nu} e^{-ipz} e^{-z/2} , \\ F_2^I(x) &= -i J_{+(\nu+1)}(x) = -i (i\lambda)^{\nu+1} e^{-ipz} e^{+z/2} , \end{aligned}$$

$$\begin{aligned}
II \quad F_1^{II}(x) &= J_{-\nu}(x) = \left(\frac{x}{2}\right)^{-\nu} = (i\lambda)^{-\nu} e^{+ipz} e^{+z/2} , \\
F_2^{II}(x) &= +i J_{-(\nu+1)}(x) = +i (i\lambda)^{-\nu-1} e^{+ipz} e^{-z/2} ;
\end{aligned} \tag{7.13}$$

here and below we use the notation

$$\lambda = \frac{\sqrt{k_1^2 + k_2^2}}{2} .$$

In the region $z \rightarrow +\infty$, $x \rightarrow +\infty i$, using the known asymptotic formula [13]

$$J_\nu(x) \sim \sqrt{\frac{2}{\pi x}} \cos \left(x - \left(\nu + \frac{1}{2} \right) \frac{\pi}{2} \right) ,$$

we get

$$\begin{aligned}
J_{+\nu}(z \rightarrow \infty) &\sim \sqrt{\frac{2}{i\pi X}} \cos i \left(X + \frac{p\pi}{2} \right) \longrightarrow \sqrt{\frac{1}{2\pi i X}} e^{+p\pi/2} e^X , \\
J_{-\nu}(z \rightarrow \infty) &\sim \sqrt{\frac{2}{i\pi X}} \sin i \left(X - \frac{p\pi}{2} \right) \longrightarrow i \sqrt{\frac{1}{2\pi i X}} e^{-p\pi/2} e^X .
\end{aligned} \tag{7.14}$$

Thus, the solutions (7.10) behave

$$z \rightarrow +\infty, \quad x = iX \rightarrow +i\infty,$$

$$\begin{aligned}
I \quad F_1^I(x) &= J_{+\nu}(x) \sim \sqrt{\frac{1}{2\pi i X}} e^{+p\pi/2} e^X , \\
F_2^I(x) &= -i J_{+(\nu+1)}(x) \sim +\sqrt{\frac{1}{2\pi i X}} e^{+p\pi/2} e^X ; \\
II \quad F_1^{II}(x) &= J_{-\nu}(x) \sim i \sqrt{\frac{1}{2\pi i X}} e^{-p\pi/2} e^X , \\
F_2^{II}(x) &= +i J_{-(\nu+1)}(x) \sim -\sqrt{\frac{1}{2\pi i X}} e^{-p\pi/2} e^X .
\end{aligned} \tag{7.15}$$

Let us consider solutions (7.11a) in Hankel's functions [13]. They are determined in terms of $J_{\pm\nu}(x)$ as follows

$$H_\nu^1(x) = +\frac{i}{\sin \nu\pi} \left(e^{-i\nu\pi} J_{+\nu}(x) - J_{-\nu}(x) \right) ,$$

$$H_\nu^2(x) = -\frac{i}{\sin \nu \pi} \left(e^{+i\nu\pi} J_{+\nu}(x) - J_{-\nu}(x) \right) . \quad (7.16)$$

From here it is easy to set behavior of the Hankel functions for small x :
 $z \rightarrow -\infty, x \rightarrow i0$,

$$\begin{aligned} H_\nu^1(x) &\sim +\frac{i}{\sin \nu \pi} \left(e^{-i\nu\pi} (i\lambda)^\nu e^{-ipz} e^{-z/2} - (i\lambda)^{-\nu} e^{ipz} e^{+z/2} \right) \\ &\sim \frac{i}{\sin \nu \pi} e^{-i\nu\pi} (i\lambda)^\nu e^{-ipz} e^{-z/2} , \\ H_\nu^2(x) &\sim -\frac{i}{\sin \nu \pi} \left(e^{+i\nu\pi} (i\lambda)^\nu e^{-ipz} e^{-z/2} - (i\lambda)^{-\nu} e^{ipz} e^{+z/2} \right) \\ &\sim -\frac{i}{\sin \nu \pi} e^{+i\nu\pi} (i\lambda)^\nu e^{-ipz} e^{-z/2} , \end{aligned} \quad (7.17a)$$

$$\begin{aligned} H_{\nu+1}^1(x) &\sim +\frac{i}{\sin(\nu+1)\pi} \left(e^{-i(\nu+1)\pi} (i\lambda)^{\nu+1} e^{-ipz} e^{+z/2} - (i\lambda)^{-\nu-1} e^{+ipz} e^{-z/2} \right) \\ &\sim \frac{-i}{\sin(\nu+1)\pi} (i\lambda)^{-\nu-1} e^{+ipz} e^{-z/2} , \\ H_{\nu+1}^2(x) &\sim -\frac{i}{\sin(\nu+1)\pi} \left(e^{+i(\nu+1)\pi} (i\lambda)^{\nu+1} e^{-ipz} e^{+z/2} - (i\lambda)^{-\nu-1} e^{+ipz} e^{-z/2} \right) \\ &\sim \frac{i}{\sin(\nu+1)\pi} (i\lambda)^{-\nu-1} e^{+ipz} e^{-z/2} . \end{aligned} \quad (7.17b)$$

Therefore the solution of the system in Hankel functions behave as follows

$z \rightarrow -\infty, x \rightarrow i0$,

$$\begin{aligned} I \quad F_1^I(x) &= H_{+\nu}^1(x) \sim \frac{i}{\sin \nu \pi} e^{-i\nu\pi} (i\lambda)^\nu e^{-ipz} e^{-z/2} , \\ F_2^I(x) &= -iH_{+(\nu+1)}^1(x) \sim \frac{-1}{\sin(\nu+1)\pi} (i\lambda)^{-\nu-1} e^{+ipz} e^{-z/2} ; \\ II \quad F_1^{II}(x) &= H_{+\nu}^2(x) \sim -\frac{i}{\sin \nu \pi} e^{+i\nu\pi} (i\lambda)^\nu e^{-ipz} e^{-z/2} , \end{aligned}$$

$$F_2^{II}(x) = -iH_{+(\nu+1)}^2(x) \sim \frac{1}{\sin(\nu+1)\pi} (i\lambda)^{-\nu-1} e^{+ipz} e^{-z/2} . \quad (7.18)$$

Behavior of the Hankel functions of $z \rightarrow +\infty$ is given by [13]

$$\begin{aligned} H_\nu^1(x) &\sim \sqrt{\frac{2}{\pi x}} \exp \left[+i \left(x - \frac{\pi}{2} \left(\nu + \frac{1}{2} \right) \right) \right] , \\ H_\nu^2(x) &\sim \sqrt{\frac{2}{\pi x}} \exp \left[-i \left(x - \frac{\pi}{2} \left(\nu + \frac{1}{2} \right) \right) \right] , \end{aligned} \quad (7.19)$$

so that

$$\begin{aligned} z \rightarrow +\infty, \quad x = iX \rightarrow +\infty i, \\ H_\nu^1(x) &\sim \sqrt{\frac{2}{i\pi X}} \exp \left[+i \left(iX - \frac{\pi}{2} \left(\nu + \frac{1}{2} \right) \right) \right] \sim \sqrt{\frac{2}{i\pi X}} e^{-p\pi/2} e^{-X} , \\ H_\nu^2(x) &\sim \sqrt{\frac{2}{i\pi X}} \exp \left[-i \left(iX - \frac{\pi}{2} \left(\nu + \frac{1}{2} \right) \right) \right] \sim \\ &\sim \sqrt{\frac{2}{i\pi X}} e^{+p\pi/2} e^{+X} , \\ H_{\nu+1}^1(x) &\sim \sqrt{\frac{2}{i\pi X}} \exp \left[+i \left(iX - \frac{\pi}{2} \left(\nu + 1 + \frac{1}{2} \right) \right) \right] \sim \\ &\sim -i \sqrt{\frac{2}{i\pi X}} e^{-p\pi/2} e^{-X} , \\ H_{\nu+1}^2(x) &\sim \sqrt{\frac{2}{i\pi X}} \exp \left[-i \left(iX - \frac{\pi}{2} \left(\nu + 1 + \frac{1}{2} \right) \right) \right] \sim \\ &\sim +i \sqrt{\frac{2}{i\pi X}} e^{+p\pi/2} e^{+X} . \end{aligned} \quad (7.20)$$

$$\begin{aligned} I \quad F_1^I(x) &= H_{+\nu}^1(x) \sim \sqrt{\frac{2}{i\pi X}} e^{-p\pi/2} e^{-X} , \\ F_2^I(x) &= -i H_{+(\nu+1)}^1(x) \sim -\sqrt{\frac{2}{i\pi X}} e^{-p\pi/2} e^{-X} ; \\ II \quad F_1^{II}(x) &= H_{+\nu}^2(x) \sim \sqrt{\frac{2}{i\pi X}} e^{+p\pi/2} e^{+X} , \end{aligned}$$

$$F_2^{II}(x) = -i H_{+(\nu+1)}^2(x) \sim \sqrt{\frac{2}{i\pi X}} e^{+p\pi/2} e^{+X} . \quad (7.21)$$

Finally, let us consider solutions in terms of the Neumann functions. These functions are as follows expressed in terms of Bessel functions [13]

$$\begin{aligned} N_\nu(x) &= \frac{\cos \nu\pi J_\nu(x) - J_{-\nu}(x)}{\sin \nu\pi} , \\ N_{-\nu}(x) &= \frac{J_\nu(x) - \cos \nu\pi J_{-\nu}(x)}{\sin \nu\pi} . \end{aligned} \quad (7.22)$$

In the region $z \rightarrow +\infty$, ($x \rightarrow +\infty i$) we use the asymptotic formulas

$$\begin{aligned} N_\nu(x) &\sim \sqrt{\frac{2}{i\pi X}} \sin \left(iX - \left(\nu + \frac{1}{2} \right) \frac{\pi}{2} \right) , \\ \sin(iX - a) &= \frac{e^{-X-ia} - e^{+X+ia}}{2} \sim \frac{i}{2} e^X e^{+ia} , \end{aligned}$$

we get

$$\begin{aligned} N_{+\nu}(z \rightarrow \infty) &\sim +i \sqrt{\frac{1}{2\pi i X}} e^{p\pi/2} e^X , \\ N_{+\nu+1}(z \rightarrow \infty) &\sim - \sqrt{\frac{1}{2\pi i X}} e^{p\pi/2} e^X , \\ N_{-\nu}(z \rightarrow \infty) &\sim \sqrt{\frac{1}{2\pi i X}} e^{-p\pi/2} e^X , \\ N_{-\nu-1}(z \rightarrow \infty) &\sim i \sqrt{\frac{1}{2\pi i X}} e^{-p\pi/2} e^X . \end{aligned} \quad (7.23)$$

Thus, the solution (7.12) in the functions of the Neumann behave at $z \rightarrow +\infty$ as follows:

$$\begin{aligned} I \quad F_1^I(x) &= N_{+\nu}(x) \sim i \sqrt{\frac{1}{2\pi i X}} e^{p\pi/2} e^X , \\ F_2^I(x) &= -i N_{+(\nu+1)}(x) \sim +i \sqrt{\frac{1}{2\pi i X}} e^{p\pi/2} e^X ; \\ II \quad F_1^{II}(x) &= N_{-\nu}(x) \sim \sqrt{\frac{1}{2\pi i X}} e^{-p\pi/2} e^X , \\ F_2^{II}(x) &= +i N_{-(\nu+1)}(x) \sim - \sqrt{\frac{1}{2\pi i X}} e^{-p\pi/2} e^X . \end{aligned} \quad (7.24)$$

Now, using the definition (7.22), we find the behavior of the Neumann functions at $z \rightarrow -\infty$.

$$\begin{aligned}
N_\nu(x) &= \frac{\cos \nu \pi J_\nu(x) - J_{-\nu}(x)}{\sin \nu \pi} \\
&\sim \frac{\cos \nu \pi (i\lambda)^\nu e^{-ipz} e^{-z/2} - (i\lambda)^{-\nu} e^{+ipz} e^{+z/2}}{\sin \nu \pi} = \\
&\sim \frac{\cos \nu \pi}{\sin \nu \pi} (i\lambda)^\nu e^{-ipz} e^{-z/2}, \\
N_{\nu+1}(x) &= \frac{\cos(\nu+1)\pi J_{\nu+1}(x) - J_{-\nu-1}(x)}{\sin(\nu+1)\pi} \\
&\sim \frac{\cos(\nu+1)\pi (i\lambda)^{\nu+1} e^{-ipz} e^{+z/2} - (i\lambda)^{-\nu-1} e^{+ipz} e^{-z/2}}{\sin(\nu+1)\pi} \\
&\sim \frac{1}{\sin(\nu+1)\pi} (i\lambda)^{-\nu-1} e^{+ipz} e^{-z/2}, \\
N_{-\nu}(x) &= \frac{J_\nu(x) - \cos \nu \pi J_{-\nu}(x)}{\sin \nu \pi} \\
&\sim \frac{(i\lambda)^\nu e^{-ipz} e^{-z/2} - \cos \nu \pi (i\lambda)^{-\nu} e^{+ipz} e^{+z/2}}{\sin \nu \pi} \\
&\sim \frac{1}{\sin \nu \pi} (i\lambda/2)^\nu e^{-ipz} e^{-z/2}, \\
N_{-\nu-1}(x) &= \frac{J_{\nu+1}(x) - \cos(\nu+1)\pi J_{-\nu-1}(x)}{\sin(\nu+1)\pi} \\
&\sim \frac{(i\lambda)^{\nu+1} e^{-ipz} e^{+z/2} - \cos(\nu+1)\pi (i\lambda)^{-\nu-1} e^{+ipz} e^{-z/2}}{\sin(\nu+1)\pi} \\
&\sim \frac{-\cos(\nu+1)\pi}{\sin(\nu+1)\pi} (i\lambda)^{-\nu-1} e^{+ipz} e^{-z/2}.
\end{aligned}$$

Consequently, the solution (7.12) in the functions of the system behave Neumann $z \rightarrow -\infty$ as follows: $z \rightarrow -\infty$, $x \rightarrow i0$,

$$\begin{aligned}
I \quad F_1^I(x) &= N_{+\nu}(x) \sim \frac{\cos \nu \pi}{\sin \nu \pi} (i\lambda)^\nu e^{-ipz} e^{-z/2}, \\
F_2^I(x) &= -i N_{+(\nu+1)}(x) \sim -i \frac{1}{\sin(\nu+1)\pi} (i\lambda)^{-\nu-1} e^{+ipz} e^{-z/2};
\end{aligned}$$

$$\begin{aligned}
II \quad F_1^{II}(x) &= N_{-\nu}(x) \sim \frac{1}{\sin \nu \pi} (i\lambda)^\nu e^{-ipz} e^{-z/2}, \\
F_2^{II}(x) &= +i N_{-(\nu+1)}(x) \sim +i \frac{-\cos(\nu+1)\pi}{\sin(\nu+1)\pi} (i\lambda)^{-\nu-1} e^{+ipz} e^{-z/2}.
\end{aligned}
\tag{7.25}$$

The results can be collected into a table.

$$\sigma = -p$$

Bessel			Hankel			Neumann		
	$z \rightarrow -\infty$	$z \rightarrow +\infty$		$z \rightarrow -\infty$	$z \rightarrow +\infty$		$z \rightarrow -\infty$	$z \rightarrow +\infty$
Z_1^I	e^{-ipz}	e^{+X}	Z_1^I	e^{-ipz}	e^{-X}	Z_1^I	e^{-ipz}	e^{+X}
Z_2^I	0	e^{+X}	Z_2^I	e^{+ipz}	e^{-X}	Z_2^I	e^{+ipz}	e^{+X}
Z_1^{II}	0	e^{+X}	Z_1^{II}	e^{-ipz}	e^{+X}	Z_1^{II}	e^{-ipz}	e^{+X}
Z_2^{II}	$e^{+i[z}$	e^{+X}	Z_2^{II}	e^{+ipz}	e^{+X}	Z_2^{II}	e^{+ipz}	e^{+X}

The most interesting are solutions of the type I in Hankel functions:

$$\begin{aligned}
z \rightarrow -\infty, \quad Z_1^I &\sim e^{-ipz}, \quad Z_2^I \sim e^{+ipz}, \\
z \rightarrow +\infty, \quad Z_1^I &\sim e^{-X} \rightarrow 0, \quad Z_2^I \sim e^{-X} \rightarrow 0,
\end{aligned}$$

which means that the problem posed in Lobachevsky space can simulate a situation in the flat space for a quantum-mechanical particle of spin $1/2$ (Dirac electron and Weyl neutrino) in a 2-dimensional potential barrier smoothly rising to infinity on the right. Electromagnetic field behaves itself similarly – see in [14].

Conclusions and acknowledgement

In the paper complete systems of exact solutions for Dirac and Weyl equations in the Lobachevsky space H_3 are constructed on the base of the method of separation of the variables in quasi-cartezian coordinates. An extended helicity operator is introduced. It is shown that solution constructed when translating to the limit of vanishing curvature coincide with common plane wave solutions on Minkowski space going in opposite z -directions. Electromagnetic field behaves itself similarly.

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